

Chapter 4

Statement

E
↑ increases
↓ decreases

Light acts as waves

Light acts as photons

Particles act as "particles"

Particles act as waves

Acceptance

✓ Common experience

✓ Photoelectric

✓ Common experience

? Some find difficult to accept.

Unifying the concepts

photons

Total energy

$E, h\nu, hc/\lambda$

Kinetic energy
momentum
Wavelength

$\frac{E}{c}, \frac{h\nu}{c}, \frac{hc}{\lambda}$

$\frac{E}{hc}, \frac{h\nu}{hc}, \frac{1}{\lambda}$

Planck

non-rel e^- 's

$mc^2 + \frac{1}{2}mv^2$

$\frac{1}{2}mv^2$

p, mv

?

rel e^- 's

$[p^2c^2 + (mc^2)^2]^{1/2}$

$[(pc)^2 + (mc^2)^2]^{1/2} - mc^2$

? p

de Broglie's unifying concept

If waves can act as particles,
particles can act as waves

$$\lambda = h/p$$

these relations hold for electrons

Some examples

	<u>mass</u>	<u> velocity / E_k</u>	<u>λ</u>
a) speeding car	1000 kg	100 m/s	$\left. \begin{array}{l} \text{"physics"} \\ \text{"everyday"} \\ \text{"small"} \end{array} \right\} \begin{array}{l} 7 \times 10^{-39} \text{ m} \\ 7 \times 10^{-34} \text{ m} \\ 7 \times 10^{-20} \text{ m} \\ 7 \times 10^{-15} \text{ m} \end{array}$
b) fired bullet	10 g	500 m/s	
c) smoke particle	10^{-9} g	1 cm/s	
d) biological cell	10^{-12} g	0.01 cm/s	
e) electron	$9.10938215(45) \times 10^{-31}$ kg	1 eV	$\left. \begin{array}{l} 1 \text{ nm (atomic size)} \\ 10 \text{ fm (nuclear size)} \end{array} \right\}$
f) electron		100 MeV	

Our concept of
small / large scale
#s is insecure

These wavelengths are
physical. When a wavelength
is of the order or larger
than the physical system it
interacts with, its wavelength
nature is evident.

How old is the universe (in sec) (3×10^{12})

Bill Gates' fortune in \$? (54×10^9)

About 24/day since the universe started

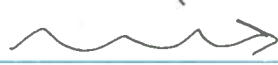
Wave superposition

In 1D, the amplitude of a wave is given by $\omega = ck$ for photons

$$A(x, t) = \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{\infty} dk A(k) e^{i(kx - \omega(k)t)} \quad (1)$$

This is
a superposition
of elemental
waves

if k is negative


if k is positive


continuous wave

→ x axis

amplitude
for a given
 k

Example

1) If

$$A(k) = A_0 \delta(k - k_0)$$

Dirac δ -function

$$A(x, t) = \frac{A_0}{(2\pi)^{1/2}} e^{i(k_0 x - \omega(k_0)t)}$$

i.e. a pure wave

→ if $k_0 > 0$

← if $k_0 < 0$

The Dirac δ function, more an idea.
Many mathematical forms.

Most important property

$$\int_a^b f(x) \delta(x - x_0) dx = f(x_0) \text{ if } a < x_0 < b$$

$$= 0 \text{ if } [x_0 < a \text{ or } x_0 > b]$$

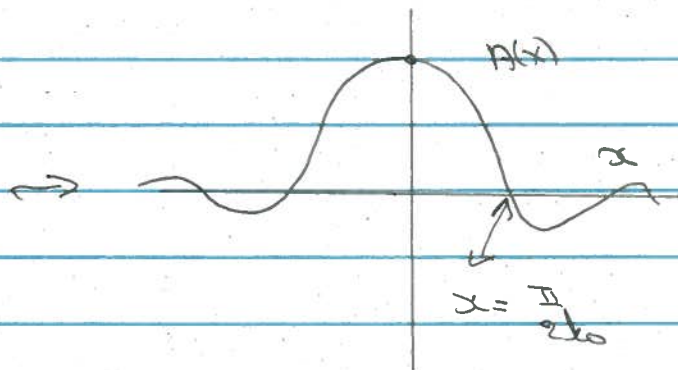
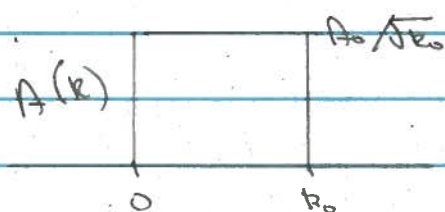
(1) mathematical form is a Fourier Transform

Some properties

$$A(x,0) = \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{\infty} dk A(k) e^{ikx} \quad (2)$$

$$A(k) = \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{\infty} dx A(x,0) e^{-ikx} \quad (3)$$

Example 2



$$A(k) = A_0 \quad \text{for } 0 \leq k \leq k_0$$

otherwise

note:
smaller k_0 is

$$A(x,0) = \left(\frac{2}{\pi}\right)^{1/2} A_0 \frac{\sin k_0 x}{x}$$

large spread
in $A(x,0)$ is

Note: (2) $\Rightarrow$ (3) $\Rightarrow$

$$\begin{aligned}
 A(k) &= \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{\infty} dx \left[\frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{\infty} dk' A(k') e^{ik'x} \right] e^{-ikx} \\
 &= \int_{-\infty}^{\infty} dk' A(k') \underbrace{\left(\frac{1}{2\pi} \int_{-\infty}^{\infty} dx e^{i(k'-k)x} \right)}_{=\delta(k'-k)} \\
 &= A(k) \quad \checkmark
 \end{aligned}$$

Another form of the δ function

$$\delta(k-k') = \frac{1}{2\pi} \int_{-\infty}^{\infty} dx e^{+i(k-k')x}$$

- very common in Fourier transforms

"Classical" wave superposition

The notes for today's lecture start at 9/25/09-3 and continue below.

Recall

$$A(x,t) = \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{\infty} dk A(k) e^{i(kx - \omega(k)t)} \quad (1)$$

$A(x,t)$ is the amplitude in (x,t) space

$A(k)$ is the " " " k space.

The "norm" of a wave is given by

$$|A|^2 = \int_{-\infty}^{\infty} dx A^*(x,t) A(x,t) \quad (2)$$

$|A(x,t)|^2$ is sometimes called the density

① → ② ⇒

$$|A|^2 = \int_{-\infty}^{\infty} dx \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{\infty} dk' A^*(k') e^{-i(k'x - \omega(k')t)} \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{\infty} dk A(k) e^{i(kx - \omega(k)t)}$$

note

$$\begin{aligned} \text{reorganizing} &= \int_{-\infty}^{\infty} dk \left(\int_{-\infty}^{\infty} dk' A^*(k') A(k) e^{i(\omega(k') - \omega(k))t} \right) \frac{1}{2\pi} \int_{-\infty}^{\infty} dx e^{i(k-k')x} \\ &= \delta(k-k') \quad (\text{from before}) \end{aligned}$$

$$= \int_{-\infty}^{\infty} dk \int_{-\infty}^{\infty} dk' A^*(k') A(k) e^{i(\omega^*(k') - \omega(k))t} \delta(k - k')$$

$$= \int_{-\infty}^{\infty} dk A^*(k) A(k) e^{i(\omega^*(k) - \omega(k))t}$$

= 0 if $\omega(k)$ is real

Note: if $\text{Im}(\omega(k)) \neq 0$
it can model
absorption or sources

$$= \int_{-\infty}^{\infty} dk |A(k)|^2$$

$$\boxed{|A|^2 = \int_{-\infty}^{\infty} dk |A(k)|^2}$$

(3)

Expectation values.

Waves are spread out over all space. Hence, the question

"Where is the wave?" is not a sensible question.

"Where is it concentrated?" is a more reasonable question.

More precise questions:

Q. What is its "average" position?

A. $\langle x \rangle = \int_{-\infty}^{\infty} dx A^*(x, t) x A(x, t)$ (4) is the answer

by convention, goes in between

Q. What is the "spread" of the wave about its average position?

A. "spread" $\equiv (\Delta x)^2 \equiv \langle x^2 \rangle - \langle x \rangle^2$

$$\langle x^2 \rangle = \int_{-\infty}^{\infty} dx A^*(x,t) \underline{x^2} A(x,t) \quad (5)$$

and $\langle x \rangle$ is given by (4)

Q. What is the average wavenumber of the wave?

A. $\langle k \rangle = \int_{-\infty}^{\infty} dx A^*(x,t) \left[\underbrace{-i \frac{\partial}{\partial x}}_{\text{operates to the right}} \right] A(x,t)$

$$= \int_{-\infty}^{\infty} dx \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{\infty} dk' A^*(k') e^{-i(k'x - \omega(k')t)} \left(-i \frac{\partial}{\partial x} \right) \int_{-\infty}^{\infty} dk A(k) e^{i(kx - \omega(k)t)}$$

$$= \int_{-\infty}^{\infty} dx \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{\infty} dk' A^*(k') e^{-i(k'x - \omega(k')t)} \int_{-\infty}^{\infty} dk A(k) \underline{k} e^{i(kx - \omega(k)t)}$$

$$= \int_{-\infty}^{\infty} dk' \int_{-\infty}^{\infty} dk A^*(k') k A(k) e^{i(\omega(k') - \omega(k))t} \underbrace{\frac{1}{2\pi} \int_{-\infty}^{\infty} dx e^{i(k-k')x}}_{\delta(k-k')}$$

$$= \int_{-\infty}^{\infty} dk |A(k)|^2 k e^{i(\omega(k) - \omega(k))t} \quad \begin{matrix} = 0 \text{ if } \omega \text{ is real} \end{matrix}$$

$$\boxed{\langle k \rangle = \int_{-\infty}^{\infty} dk k |A(k)|^2} \quad (6)$$

Q: What is the "spread" of the wavenumbers about the average.

A: "Spread" $\equiv (\Delta k)^2 \equiv \langle k^2 \rangle - \langle k \rangle^2$

$$\langle k^2 \rangle = \int_{-\infty}^{\infty} dx A^*(x,t) \left[-i \frac{\partial}{\partial x} \right]^2 A(x,t) = \int_{-\infty}^{\infty} dk k^2 |A(k)|^2 \quad (7)$$

"Uncertainty" relationship for general waves

It can be proven that:

$$(\Delta k)^2 (\Delta x)^2 \geq \frac{1}{4}$$

Interpretation:

If you make a wavepacket, the tighter Δx is, the greater the spread in k . Conversely, the tighter Δk is, the more widespread the function in x -space is.

Aside: To prove apply the Cauchy-Schwarz inequality

$$\left| \int dx f(x) g(x) \right|^2 \leq \left(\int dx |f(x)|^2 \right) \left(\int dx |g(x)|^2 \right)$$

where f, g are any square integrable complex functions (See Wikipedia!)

Example of a wave packet

$$\text{Let } A(k) = \alpha^{1/2} e^{-\alpha|k|} \quad (1)$$

$$\begin{aligned} \text{norm } |A|^2 &= \int_{-\infty}^{\infty} dk A^*(k) A(k) \quad (\text{general}) \\ &= \alpha \int_{-\infty}^{\infty} dk e^{-2\alpha|k|} \quad (\text{this example}) \\ &= 2\alpha \int_0^{\infty} dk e^{-2\alpha k} \quad (\text{used symmetry}) \\ &= \int_0^{\infty} dz e^{-z} \quad (z = 2\alpha k) \\ &= 1 \quad \checkmark (\text{this, by design}) \end{aligned}$$

$$\begin{aligned} \langle k \rangle &= \int_{-\infty}^{\infty} dk k |A(k)|^2 \quad (\text{general}) \\ &= 0 \quad (\text{antisymmetry}) \end{aligned}$$

$$\langle k \rangle = 0 \quad (2)$$

← Mathematica or
Mathlab code is posted

(*)

Integrate [z^2 Exp[-z], {z, 0, Infinity}]

$$\langle k^2 \rangle = \int_{-\infty}^{\infty} dk k^2 |A(k)|^2 \quad (\text{general})$$

$$= 2 \int_{-\infty}^{\infty} dk k^2 e^{-2\alpha|k|} \quad (\text{this example})$$

$$= 2\alpha \int_{-\infty}^{\infty} dk k^2 e^{-2\alpha k} \quad (\text{symmetry})$$

$$= \frac{1}{4\alpha^2} \int_{-\infty}^{\infty} dz z^2 e^{-z} \quad (z = 2\alpha k)$$

= 2, Mathematica

$$= \frac{1}{2\alpha^2}$$

$$\langle k^2 \rangle = \frac{1}{2\alpha^2} \quad (3)$$

Find $A(x_0)$

$$A(x_0) = \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{\infty} dk A(k) e^{ikx} \quad (\text{general})$$

$$\left(\frac{\alpha}{2\pi}\right)^{1/2} \int_{-\infty}^{\infty} dk e^{-\alpha|k|} \underbrace{e^{ikx}}_{(\cos kx + i \sin kx)} \quad (\text{this example})$$

$$= \left(\frac{\alpha}{2\pi}\right)^{1/2} \int_{-\infty}^{\infty} dk e^{-\alpha|k|} \cos kx \quad (\text{antisymmetry of } \sin kx \text{ term})$$

$$= \left(\frac{2\alpha}{\pi}\right)^{1/2} \int_0^{\infty} dk e^{-\alpha k} \cos kx \quad (\text{symmetry})$$

Integrate $1/(1+z^2)^2$, ξ , η , ζ , η , ζ , η , ζ

$$A(x,0) = \left(\frac{2a}{\pi} \right)^{1/2} \frac{a}{a^2+x^2} \quad (4)$$

Check norm

$$|A|^2 = \int_{-\infty}^{\infty} dx |A(x,0)|^2$$

$$= \frac{2a}{\pi} \int_{-\infty}^{\infty} dx \frac{a^2}{(a^2+x^2)^2}$$

$$= \frac{2}{\pi} \int_{-\infty}^{\infty} dz \frac{1}{(1+z^2)^2} \quad x = za$$

$$|A|^2 = 1$$

$$\langle x \rangle = 0 \quad (5) \quad \text{by antisymmetry}$$

$$\langle x^2 \rangle = \int_{-\infty}^{\infty} dx A^*(x,0) x^2 A(x,0) \quad (\text{general})$$

$$= \left(\frac{2a}{\pi} \right) \int_{-\infty}^{\infty} dx x^2 \frac{a^2}{(a^2+x^2)^2}$$

$$= \frac{2a^2}{\pi} \int_{-\infty}^{\infty} dz \frac{z^2}{(1+z^2)^2} \quad x = za$$

$$= a^2$$

(*)

means
Mathematica
or Matlab
code is
posted

(*)

$$\langle x^2 \rangle = \alpha^2 \quad (6)$$

$$\text{So, } \langle x \rangle = \langle k \rangle = 0$$

$$\langle x^2 \rangle = \alpha^2 \quad \langle k^2 \rangle = \frac{1}{2\alpha^2}$$

$$(\Delta k)^2 (\Delta x)^2 = (\langle k^2 \rangle - \langle k \rangle^2) (\langle x^2 \rangle - \langle x \rangle^2)$$

$$= \frac{1}{2} \geq \frac{1}{4} \text{ , as expected}$$

$A(x, t)$?

$$A(x, t) = \frac{1}{(2\pi)^{1/2}} \int_a^b dk A(k) e^{i(kx - \omega t)} \quad (\text{general})$$

$$= \left(\frac{\alpha}{2\pi} \right)^{1/2} \int_a^\infty dk e^{-\alpha|k|} e^{i(kx - c|k|t)} \quad (\text{this example})$$

Note: For light waves

$$\omega = c|k|$$

$$A(x,t) = \left(\frac{\alpha}{2\pi}\right)^{1/2} \left\{ \int_{-\infty}^0 dk e^{-\alpha k} e^{ikx} e^{-ickt} + \int_0^{\infty} dk e^{-\alpha k} e^{ikx} e^{-ickt} \right\}$$

$$= \left(\frac{\alpha}{2\pi}\right)^{1/2} \left\{ \int_0^{\infty} dk e^{-\alpha k} e^{-ikx} e^{-ickt} + \int_0^{\infty} dk e^{-\alpha k} e^{ikx} e^{-ickt} \right\} \quad \begin{array}{l} k \rightarrow -k \\ \text{in 1st} \\ \text{integral} \end{array}$$

$$= \left(\frac{2\alpha}{\pi}\right)^{1/2} \int_0^{\infty} dk e^{-\alpha k} \cos kx e^{-ickt}$$

$$(*) \quad A(x,t) = \left(\frac{2\alpha}{\pi}\right)^{1/2} \frac{\alpha + ickt}{(\alpha + ickt)^2 + x^2} \quad (7)$$

This represents a packet that is spreading out with time

Demonstration?

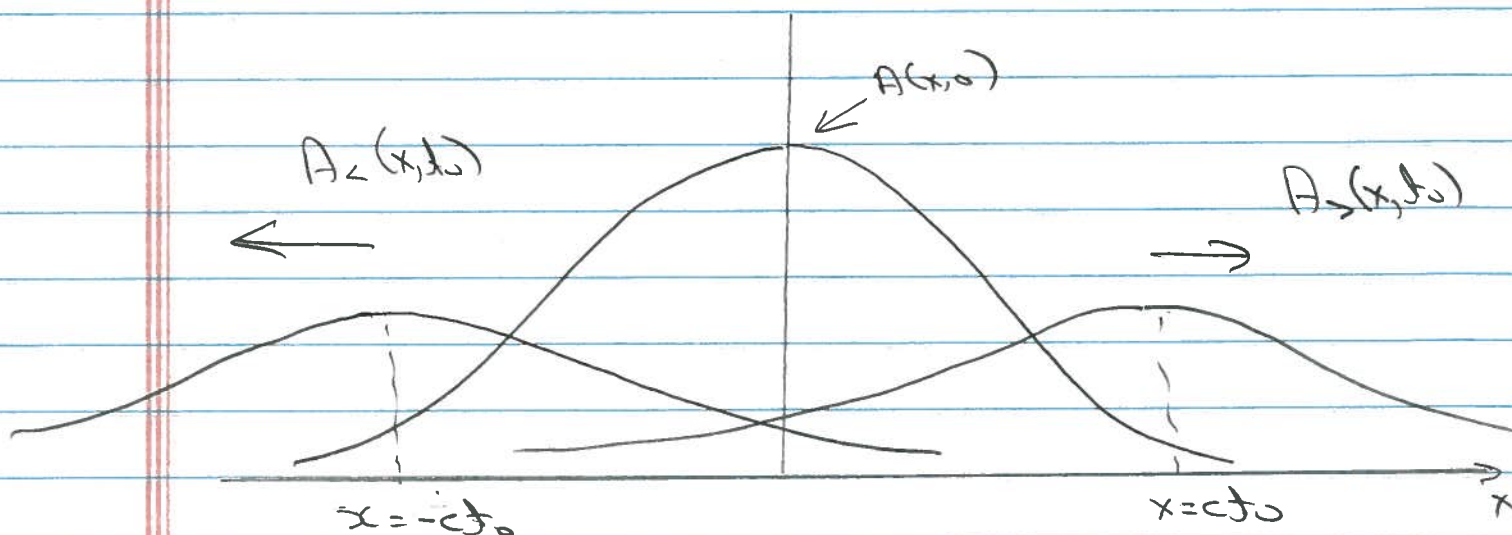
Interpretation

Note that (7) can be rewritten

$$(8) \quad A(x,t) = \left(\frac{\alpha}{2\pi}\right)^{1/2} \left\{ \overbrace{\frac{1}{\alpha - i(x-ct)}}^{A_+(x,t)} + \frac{1}{\alpha + i(x+ct)} \right\}$$

↑
this part is a wavepacket
travelling with velocity
 $\vec{v} = c\hat{x}$, centered
at $x=ct$, that
arises from
the positive
 k 's

↑
this arises
from the
negative
 k 's



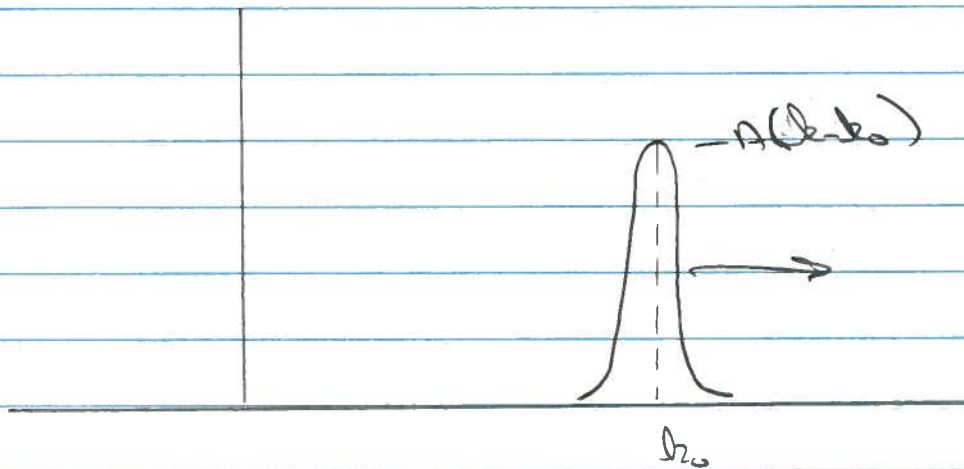
Note that the shape of the A_+ & A_- wave packets does not change with time, this is a consequence of all components in k having the same speed, c .

How does a wave packet move?

(9) Recall
$$\psi(x,t) = \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{\infty} dk A(k) e^{i(kx - \omega(k)t)}$$

This is a general expression, for any kind of wave, where $\omega(k)$ is any function of k .

Imagine that $A(k)$ is tightly constrained in the vicinity of k_0 .



In fact, it is so tightly constrained, that we only need to consider ω in the vicinity of k_0 .

That is

(61)

$$\omega(k) \rightarrow \omega(k_0) + \left. \frac{d\omega}{dk} \right|_{k_0} (k - k_0) + \text{terms}$$

we

can

~~ignore~~

and

$$A(x, t) = \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{\infty} dk A(k - k_0) \exp \left\{ i(k - k_0 + k_0)x - i \left(\omega(k_0) + \left. \frac{d\omega}{dk} \right|_{k_0} (k - k_0) \right) t \right\}$$

$$= \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{\infty} dk A(k - k_0) \exp \left\{ i(k - k_0) \left(x - \left. \frac{d\omega}{dk} \right|_{k_0} t \right) \right\} + e^{i(k_0 x - \omega(k_0) t)}$$

$$= \left\{ \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{\infty} dk A(k) e^{ik(x - \frac{d\omega}{dk} t)} \right\} e^{i(k_0 x - \omega(k_0) t)}$$

Shift
integration
variable

$$= \underbrace{A \left(x - \left. \frac{d\omega}{dk} \right|_{k_0} t \right)}_{\text{Coherent wave packet moving with velocity}} \underbrace{e^{i(k_0 x - \omega(k_0) t)}}_{\text{pure wave monochromatic arbitrary phase}}$$

Coherent wave

packet moving with

velocity

$$v_g = \left. \frac{d\omega}{dk} \right|_{k=k_0}$$

Does it make sense?

light

$$\omega = kc$$

$$\frac{d\omega}{dk} = c \leftarrow \text{group velocity} \checkmark$$

particles with mass

$$v_g = \frac{d\omega}{dk} = \frac{m d\omega}{m dk} = \frac{dE}{dp}$$

$v_g = \frac{dE}{dp}$, another general expression
Since $E = \hbar\omega$ for light,
it works for light too.

non-relativistic particles

$$E = \frac{p^2}{2m}$$

$$v_g = \frac{dE}{dp} = \frac{p}{m} = v \quad \checkmark$$

relativistic particles

$$\begin{aligned} E^2 &= c^2 p^2 + (mc^2)^2 \\ 2E \frac{dE}{dp} &= 2c^2 p \end{aligned} \quad \left. \begin{aligned} & \\ & \end{aligned} \right\} \begin{aligned} v_g &= \frac{dE}{dp} = \frac{pc^2}{E} \\ &= \frac{mv^2 c^2}{mc^2} \\ &= v \quad \checkmark \end{aligned}$$